Local behavior of the Taylor method for stiff ODEs

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Joan Gimeno
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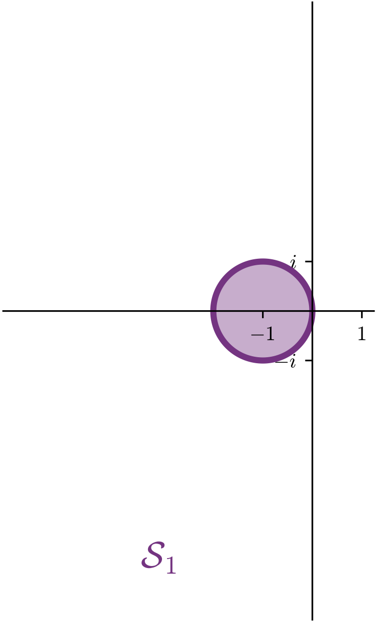

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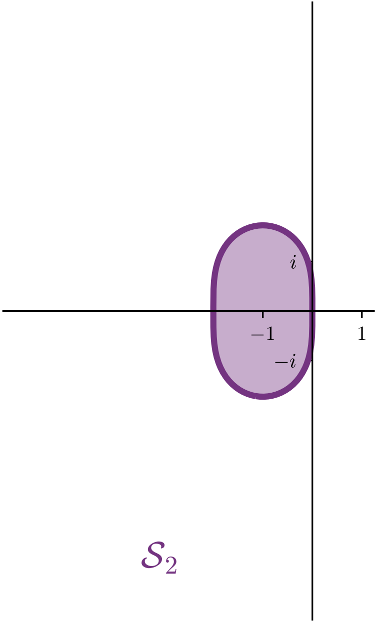

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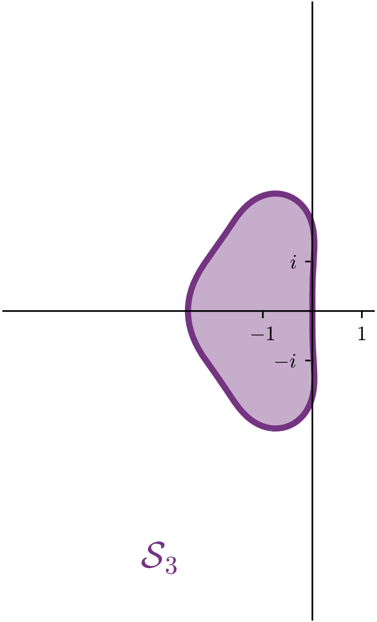

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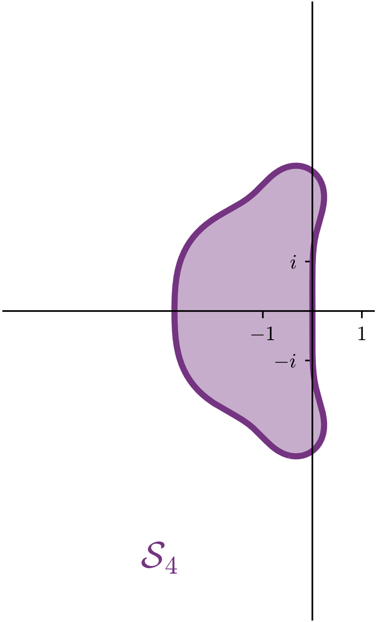

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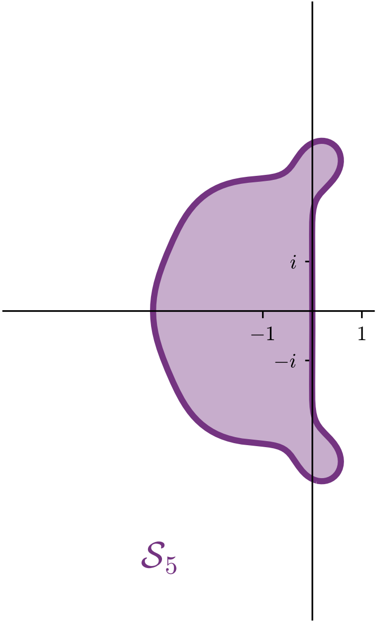

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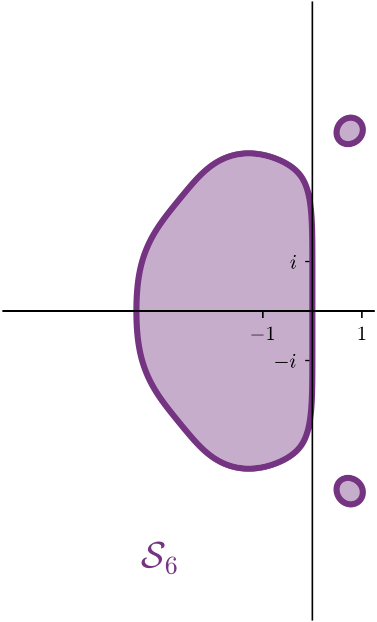

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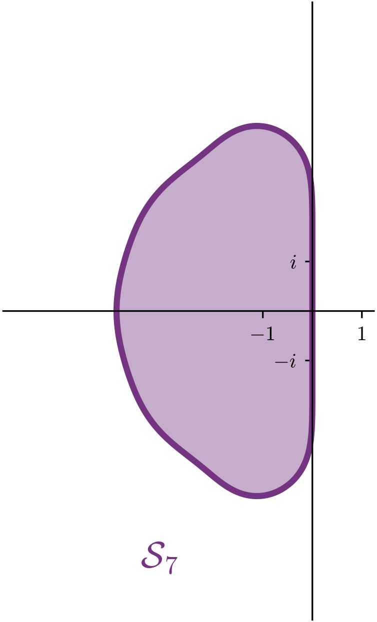

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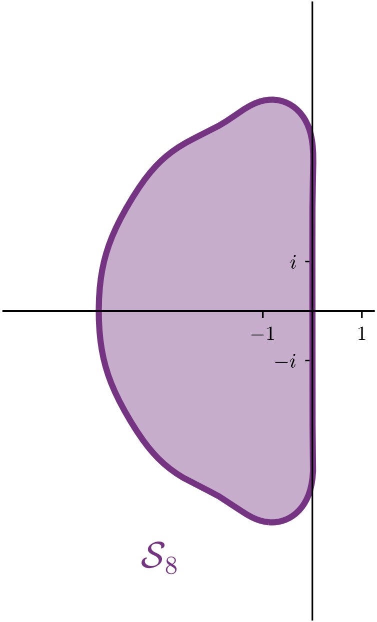

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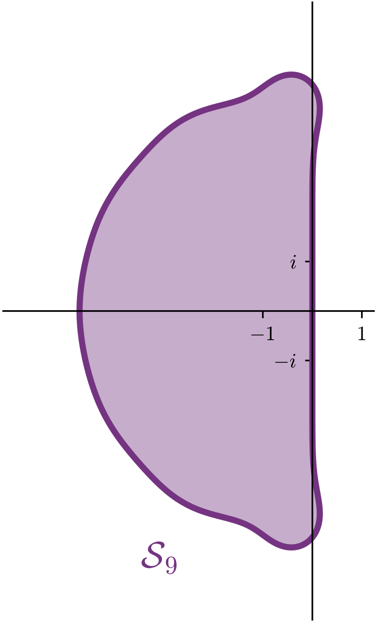

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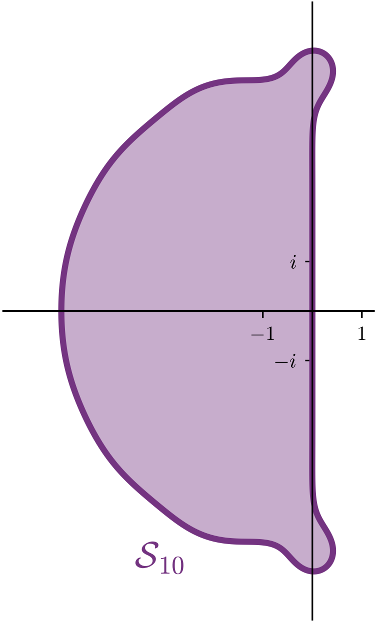

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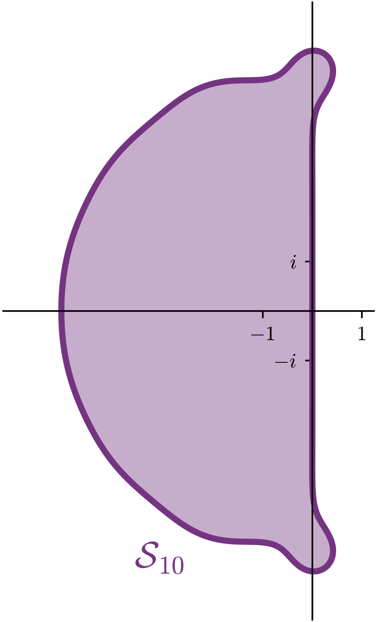

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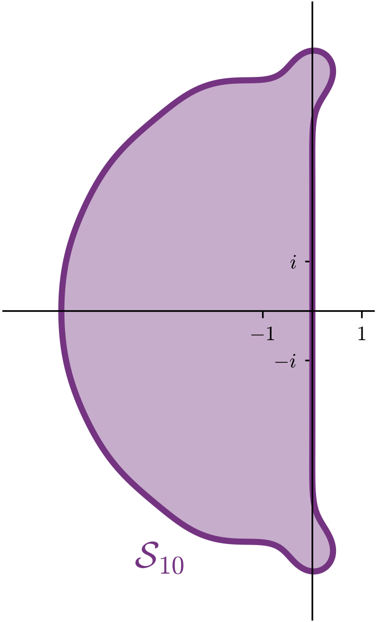

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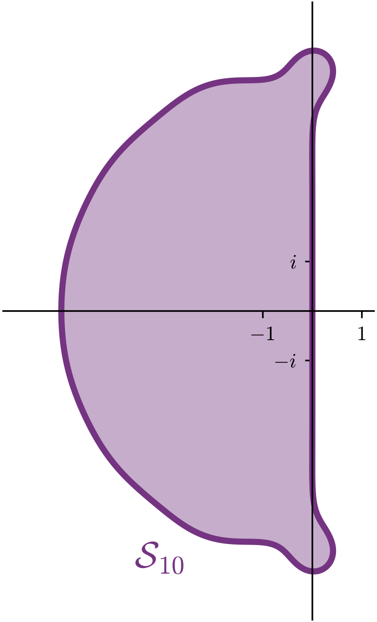

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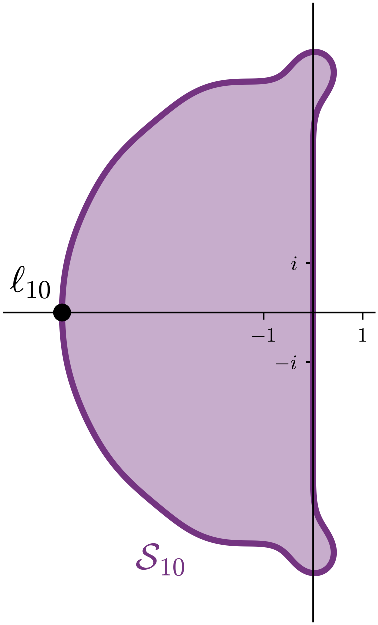

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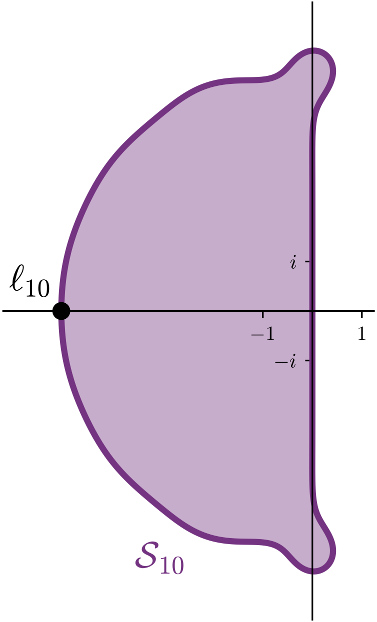

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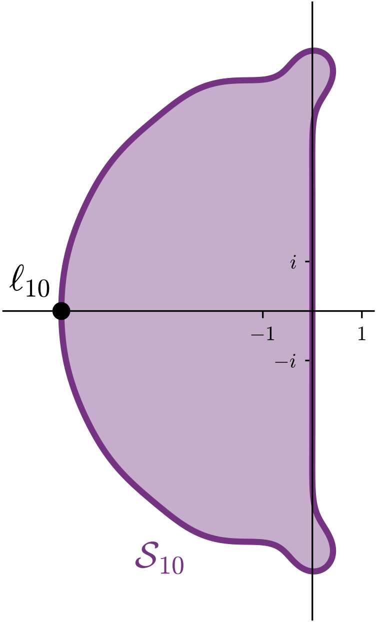

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The stiffness indicator of Söderlind *et al.* is

$$\sigma[A] = \tfrac{1}{2}(M[A] - M[-A])$$

where $A = D_y f(x, y(x))$

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The truncation error of the Taylor method is affected by

$$\left\| \sum_{k=0}^{n+1} {y_0}^{[k]} - \sum_{k=0}^{n} {\widehat{y}_0}^{[k]} \right\| \leq \left\| {y_0}^{[n+1]} \right\| h^{n+1} + \varepsilon |s_n(h \cdot \sigma[D_y f])|$$

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